JOURNAL OF MATERIALS SCIENCE 17 (1982) 994–1002

Static fatigue of glass: functional dependence of failure time on stress

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Static fatigue of as-received Pyrex borosilicate glass was examined over a wide range of stress and failure time under constant temperature, relative humidity, and surface treatment. It was necessary to calculate the load on the sample from strain guage measurements. The mean log failure time was related to the reciprocal of stress or the reciprocal of stress squared; power law and direct proportionality to stress gave poorer fits.

1. Introduction

Prediction of the life of brittle materials is important in many applications, especially when the parts are of high value or are inaccessible, as for glass fibres used as optical wave guides. Life prediction beyond experimental times requires knowledge of the functional dependence of failure time on the applied stress. This functional dependence is often assumed to be a power law, but there has been no demonstration that this dependence is correct for extrapolation to long times.

The goal of this work was to determine the functional dependence of failure time on stress for glass. Delayed failure under a static load (static fatigue) was chosen as the method most closely related to long-term failure of glass parts in practical applications. It is possible to measure strengths as a function of loading rates as a measure of fatigue, but it is difficult to make such measurements over a wide range of failure stresses.

We chose Pyrex borosilicate glass (Corning 7740) for this study because it is readily available as rods in large quantities, and has not been studied as intensively as soda-lime and silica glasses. Large numbers of samples are needed because of the spread in failure times even when all conditions are held constant. We tested more than 1200 similar samples in the same relative humidity.

During this work several different types of apparatus were used to apply a static load to the glass rods in four-point bending. The load at the bending jig was calculated from the weight hung onto a lever arm, and the arm ratio. Statistically significant differences of failure times were found for different designs. When the stress was calculated from direct measurements of strain with strain guages, a different stress from that calculated from the lever arm was found, and results from the various designs did not suffer significantly.

Several different relationships between failure time, t, and applied stress, S, in brittle materials have been proposed; they are:

linear [1],

$$\log_{10} t = a - b\left(\frac{S}{S_N}\right), \tag{1}$$

power law [2],

$$\log_{10} t = c - n\log\left(\frac{S}{S_N}\right), \tag{2}$$

inverse exponential [3–5],

$$\log_{10} t = d + g\left(\frac{S_N}{S}\right), \tag{3}$$

and inverse exponential squared [6],

$$\log_{10} t = h + p\left(\frac{S_N}{S}\right)^2. \tag{4}$$

In these equations S_N is the failure stress at liquid nitrogen temperature (~ 196° C) and a, b, c, d, g, h, n and p are constants. These equations were tested directly by least-squares fits, by examining the variation of spread in $\log_{10}$ failure times, and by their ability to predict long-time tests.

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2. Experimental methods

Glass rods 3 mm in diameter and 60 mm long were cut from as-received rods of Corning 7740 Pyrex borosilicate glass, all from the same shipment. The rods were stressed in four-point jigs with either knife-edges or rods. Tests at liquid nitrogen temperature were carried out in an Instron testing machine. In the static fatigue tests the jig was attached to a lever arm with a metal cable. Different kinds of multiple-lever systems were used. The amplification factor was calculated by simple mechanics, and a correction was made for the weights of the lever arm portions [7]. The maximum stress, S, between the linear load points was calculated from the simple bending formula:

$$S = \frac{8PL}{\pi D^3}, \tag{5}$$

where P is the load on the jig, L is the distance between the inner and outer points (19 mm) and D is the diameter of the rod sample (3 mm).

Mean fatigue times at the same nominal stress in different instruments were up to two orders of magnitude different as shown in Fig. 1, and the difference was highly significant statistically. This difference occurred because the calculated load at the jig was not quite correct. To find the correct

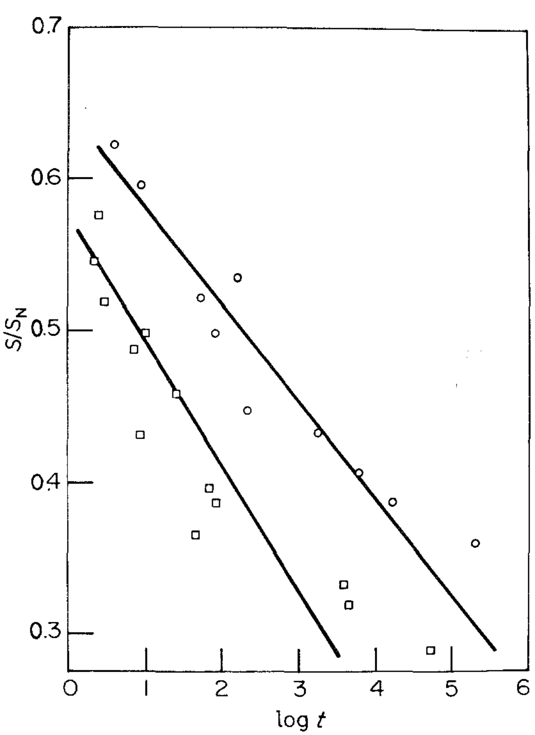

Figure 1 Mean $\log_{10}$ failure times as a function of reduced stress S/S_N for two different apparatus designs.

stress two strain gauges were mounted 180° apart on a steel bar 3.2 x 3.2 x 64 mm in dimension and wired in the half-bridge arrangement shown in Fig. 2. This arrangement was read on a Vishay/Ellis-10 strain gauge indicator. The force, P, was calculated from the equation

$$P = \frac{\epsilon E w h^2}{6L}, \tag{6}$$

where ϵ is the strain, E is Young's modulus (210 GPa for the steel used), w is the width of the beam, h is its length, and L is the distance between the outer and inner load points.

The time of loading was measured with micro-switches on the lever arms, wired to digital clocks. The clocks started when the load was applied, and when the sample broke, the lever arm hit the switch, stopping the clock.

The lever arms were entirely enclosed in plexiglass boxes and the relative humidity was controlled at 60%, as read from wet and dry bulb thermometers, by bubbling air through a saturated lithium chloride solution.

Glass and other brittle materials are often abraded before strength or fatigue testing to give more uniform strengths, even though mean strength is reduced. However, we found that abrading glass can also give strength distributions that fit distribution functions such as the normal or Weibull less well than as-received glass [8]. It is also difficult to abrade samples uniformly, and abrasion of a large number of samples takes much time. Therefore we chose to use the glass rods as-received, without any treatment. The mean strength and coefficient of variation (standard deviation divided by the mean) at − 196° C did not change significantly for samples tested throughout this work.

3. Statistical treatment of data

Strengths of brittle materials for samples that are nominally identical and are tested under identical conditions show considerable spread. These strength distributions can be described by various

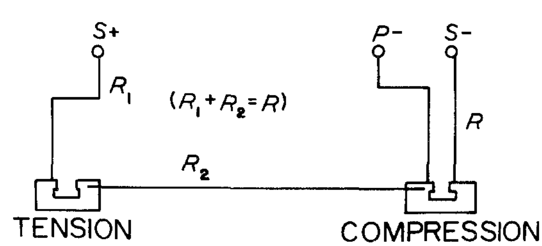

Figure 2 Wiring of strain gauges.

distribution functions. Strengths are often distributed symmetrically, so they can be described by a normal (Gaussian) distribution,

$$P = \frac{1}{d\sqrt{2\pi}} \exp\left[-\frac{(S-S_m)^2}{2d^2}\right], \qquad (7)$$

where P is the probability of finding a sample of strength (failure stress) S, S_m is the strength of greatest probability ($P_m = 1/d\sqrt{2\pi}$) and d is a measure of the spread of the distribution called the standard deviation and is equal to the root mean square of deviations from the mean strength. The integral of Equation 7 gives the fraction, F, of samples that break below the stress, S, i.e.,

$$F = \tfrac{1}{2}\{1 + \text{erf}[(S-S_m)/\sqrt{2}d]\}, \qquad (8)$$

where erf, the error function, is defined by

$$\text{erf}(X) = \frac{2}{\sqrt{\pi}} \int_0^X \exp(-\lambda^2)\,d\lambda. \qquad (9)$$

Recently, another distribution function called the Weibull distribution has become quite popular in describing strength distributions:

$$F = 1 - \exp[-(S/S_0)^m], \qquad (10)$$

where S_0 is a scaling factor and m is a measure of the spread of the distribution; the smaller m is, the broader the distribution. For a value of m larger than about three, the Weibull distribution is nearly symmetrical and closely resembles the normal distribution. A convenient way to examine the fit of the strength data to a Weibull distribution and to calculate m is to transform Equation 10 to

$$\log[-\ln(1-F)] = m \log S - m \log S_0. \qquad (11)$$

A least-squares fit of the related equation gives values of m and S_0.

A measure of fit to a function is the sum of squares

$$1 - \frac{\Sigma_{i=1}^n (S_i - \hat{S}_i)^2}{\Sigma_{i=1}^n (S_i - \bar{S})^2}. \qquad (12)$$

In this equation S_i is a measured value, $\bar{S}$ is the mean of measured values, and $\hat{S}_i$ is the predicted value of S calculated from a certain function and its estimated parameters, corresponding to the rank position of S_i (value of F). For a straight line function Equation 12 is equal to R^2, which is usually the correlation coefficient. The parameters for a straight line are the slope and intercept. For a normal distribution the parameters are the mean, S_m, and standard deviation, d, and $\hat{S}_i$ can be calculated from Equation 8 for each F. If the fit is perfect, $R^2 = 1$. If the sum of squares is greater than about 0.95, the fit is quite good; between 0.9 and 0.95 reasonably good, and below 0.9 successively poorer.

It is found for fatigue data that $\log_{10}$ failure times fit the distribution functions much better than failure times directly, so for fatigue data $\log_{10} t$ is used in place of S in Equations 7 and 8 including $\overline{\log_{10} t}$ for S_m.

4. Experimental results

There is no fatigue at $-196°$ C, so fracture strengths at this temperature give a measure of the flaw intensity and distribution undisturbed by fatigue effects. A total of 335 samples were tested at $-196°$ C including samples tested by Malitoris [9]. These samples were broken at different strain rates; a sum-of-ranks test [10] showed no statistically significant difference between samples tested at different strain rates and at different times throughout the work. The mean strength for the 335 samples was 237 MPa with a coefficient of variation of 23.9%.

Samples were annealed at 550° C for 15 and 30 minutes and tested at $-196°$ C; the results are summarized in Table I. The annealing temperature of Corning 7740 glass is about 555° C, at which temperature nearly all residual stress is removed in about 15 min. Annealing of stress, even at temperatures much below the annealing temperature, is non-linear with most of the stress annealing rapidly [11]. Thus it appears that there was

TABLE I Strength of as-received and annealed Pyrex borosilicate glass at $-196°$ C

Condition	No. of samples	Mean failure stresses (MPa) [ksi]	Coefficient of variation (%)	Weibull distribution from Equation 10	
				Slope	R^2
As-received	335	236.7 [34.3]	23.9	4.99	0.988
15 min anneal	175	223.8 [32.4]	26.1	4.15	0.929
30 min anneal	123	212.4 [30.8]	29.9	4.40	0.972

TABLE II Static fatigue of Pyrex borosilicate glass

Reduced stress, S/S_N	No. of samples	$\overline{\log_{10} t_f}$ (t_f in sec)	Standard deviation, d	Sum of squares measure of fit to normal distribution
0.329	5	5.317	1.179	0.903
0.337	13	4.811	1.177	0.950
0.352	31	4.261	1.106	0.970
0.371	53	3.744	1.484	0.959
0.389	48	3.573	1.540	0.945
0.394	65	3.214	1.634	0.964
0.408	50	2.374	1.300	0.986
0.425	44	1.740	1.253	0.970
0.435	94	2.373	1.496	0.967
0.452	94	1.898	1.028	0.982
0.466	50	1.829	1.337	0.980
0.471	50	1.731	1.065	0.964
0.489	54	2.182	1.239	0.961
0.504	21	0.924	0.863	0.950
0.531	58	1.102	0.902	0.969
0.542	57	1.296	0.994	0.967
0.568	59	0.887	0.995	0.956
0.583	99	1.040	0.845	0.980
0.608	75	0.447	0.882	0.941
0.647	50	0.289	0.783	0.869
0.673	50	0.390	0.715	0.911
0.728	50	0.530	0.629	0.931

little or no effect of residual stress on the strength of the samples listed in Table I, and that the decrease in strength probably resulted from some other factor, possible an increase in the degree of phase separation. Properties of Pyrex borosilicate glass such as viscosity [12] and chemical durability [13] change substantially during heating for as little as an hour at 600° C, as a result of phase separation. Therefore it seemed unnecessary and perhaps even damaging to anneal the specimens, so they were tested in static fatigue without annealing.

The results of mean $\log_{10}$ failure time as a function of reduced stress are summarized in Table II. The stresses were calculated from the strain gauge measurements, as described in the experimental methods section, and the mean S_N value was 237 MPa (Table I). At each stress the number of samples tested, the standard deviation of $\log_{10} t_f$ values, and the sum of squares measure of fit (Equation 11) are given in Table II. In most sets of data the measure to fit the normal distribution is quite good (< 0.95), justifying the use of $\log_{10}$ failure time as the dependent variable. Examples of a normal plot (on probability paper, Equation 8) and a Weibull plot are given in Figs 3 and 4.

In four of the data sets at the highest stresses there were an appreciable number of short fracture times (less than two seconds) that were difficult to measure reliably. An indication of this problem is the somewhat lower measure of fit to the normal distribution. In order to calculate a more reliable $\overline{\log_{10} t}$ value at these stresses the longer failure times were plotted on probability paper, and the value of $\log_{10} t$ at 50% of samples failed was taken

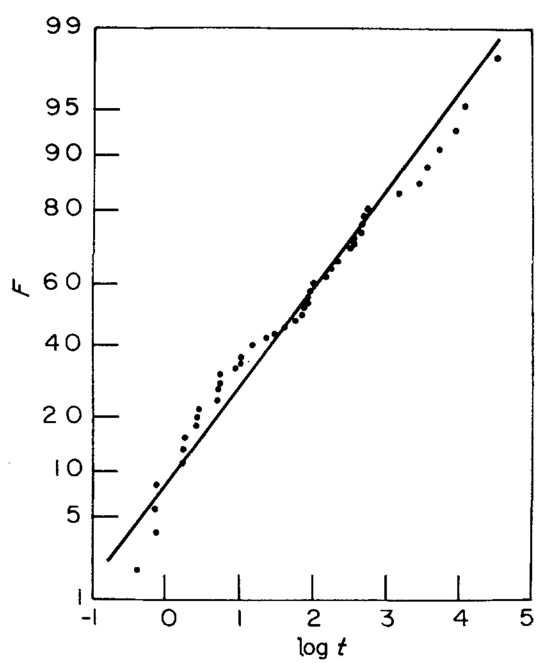

Figure 3 Fraction of samples, F, having a $\log_{10} t$ less than the abscissa value (probability plot) for $S/S_N = 0.425$. The measure of fit = 0.970.

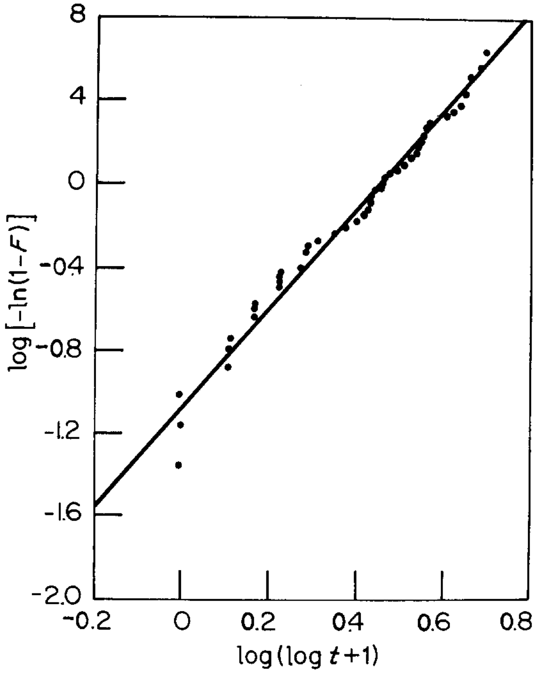

Figure 4 Weibull plot for $S/S_N = 0.425$. $R^2 = 0.978$.

as the mean value. The results are given in Table III. These values are all somewhat lower than those in Table II; however, substitution of the revised values for the original ones did not change appreciably the fits or parameters described below.

The means of $\log_{10} t_f$ values were tested for their functional dependence on stress. The mean of a sample of experimental measurements taken from a large population (for example, the $\log_{10}$ failure times for a series of samples held at a particular stress) is the "best" statistical measure of the "middle" of the distribution. It is the expected value, and is unbiased and consistent (see statistical books such as i.e. [14] and [15], for further discussion of these matters). The mean of the sample is therefore the best statistical estimate of the mean of the ficticious population of all $\log_{10} t$ (the "true" mean). The distribution of the sample means has a standard deviation $\sigma/\sqrt{n}$, where σ is the standard deviation of the population and n the number of measurements in the sample. The standard deviation, d, of the sample is the best measure of the standard deviation of the population, so that $d/\sqrt{n}$ is a measure of the reliability of the mean value calculated for the sample.

In analyzing static fatigue data some experimenters have thought that reliance on the mean of the $\log_{10} t$ values at a particular stress "makes inefficient use of the data since only the median t_f values are used", giving large uncertainties in calculated fatigue parameters [16], or that "new" information can be found by treating the data differently [17]. The discussion of the last paragraph shows that these ideas are based on a misconception. The mean value makes the most efficient use of the data, and no "new" information can be found by treating the data differently. In fact these various procedures [16–18], including the use of a "homologous" stress ratio, requires explicit or implicit assumptions about the functional dependence or failure time on stress and on the relation between the distribution of inert strengths and the distribution of $\log_{10} t$ values. Introduction of these assumptions causes a biasing or weighting of the experimental values, leading to an apparently different functional dependence of $\log_{10} t$ on stress than found from the means. Thus these procedures are not only unnecessary but they also distort the data.

TABLE III Revised values of $\overline{\log_{10} t}$

S/S_N	0.728	0.673	0.647	0.608
$\overline{\log_{10} t}$	0.30	0.23	0.20	0.26

Another misunderstanding is that the spread (standard deviation) of $\log_{10} t$ values at a particular stress is a measure of the reliability of the mean value. The true measure of the reliability of the mean is $d/\sqrt{n}$, the standard deviation of the sample divided by the square root of the number of measurements; thus the value of the mean becomes more reliable (has a high probability of being close to the population mean) as more measurements are taken, as it must. Therefore the mean values are the correct ones to use for finding the functional dependence of failure time on stress.

The mean $\log_{10} t$ values were fitted to each of Equations 1 to 4 by linear regression (least squares) analysis. The results are summarized in Table IV. The fits were first made by using the $\overline{\log_{10} t}$ and S/S_N values in Table II directly. However, because there are different numbers of samples at each of the stresses, a weighted least squares analysis is more appropriate. In this method each $\overline{\log_{10} t}$ value is multiplied by a weighting factor, w_i, which is proportional to $\sqrt{n}$, where n is the number of samples at a particular stress; $\sqrt{n}$ is used because it is a measure of the reliability of the $\overline{\log_{10} t}$ (see above). Equations for the weighted least squares calculations [20] are given in the Appendix. The calculations of fits were done on a computer; programs for the calculations are given in [7].

TABLE IV Fit of $\overline{\log_{10} t_f}$ as a function of S/S_N for Equations 1 to 4

$\overline{\log_{10} t_f}$	Equation	Least squares			Weighted average least squares		
		Slope	Intercept	R^2	Slope	Intercept	R^2
S/S_N	1	11.78	7.82	0.802	10.79	7.21	0.805
$\log_{10}(S/S_N)$	2	13.97	−2.43	0.873	12.90	−2.15	0.869
S_N/S	3	2.94	−4.26	0.925	2.78	−3.95	0.915
$(S_N/S)^2$	4	0.673	−1.20	0.955	0.655	−1.12	0.944

The results in Table IV show that there is not much difference between the ordinary and weighted regression parameters. Equation 4 (squared reciprocal) fits the data best, followed by Equation 3 (reciprocal stress, Equation 2 (power law) and Equation 1 (proportional to stress). The fit to Equation 1 is quite poor, to Equation 2 rather poor, to Equation 3 reasonably good, and to Equation 4 quite good. Plots of the fits are given in Figs 5 to 8, comparing the mean $\overline{\log_{10} t}$ values with the regression lines. The relative goodness of fit which improves in the sequence Equations 1, 2, 3 and 4, is apparent from the graphs.

Another way to judge the applicability of an equation to a set of data is to examine the residuals (see [20], Chapter 3). The residuals for the fits of Figs 5 to 8 are the differences between the measured values of $\log_{10} t$ and the values of $\log_{10} t$ calculated from the regression parameters (Table IV); the residuals can be estimated from the figures by the difference in $\log_{10} t$ values between the points and the lines in the figures. A systematic variation of residuals with either the independent or dependent variable shows that the equation chosen for fitting is inadequate. The residuals from Figs 5 and 6 (Equations 1 and 2) clearly show a systematic deviation in that the residuals are almost all positive at the higher and lower stresses or $\log_{10} t$ values and negative in between. For Figs 7 and 8 (Equations 3 and 4) there is perhaps a small tendency of this sort, but it is much less marked than for the other two figures.

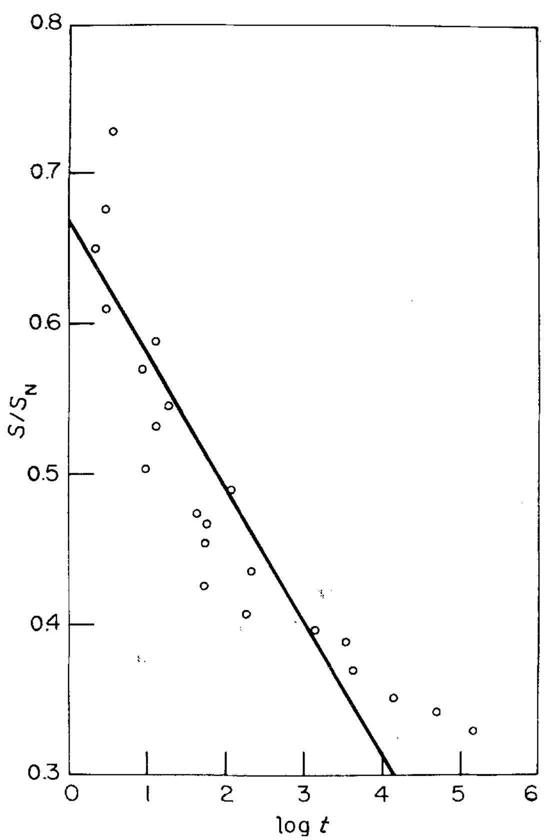

Figure 5 Fit of Equation 1 to mean $\log_{10}$ failure times of Pyrex borosilicate glass at different stresses.

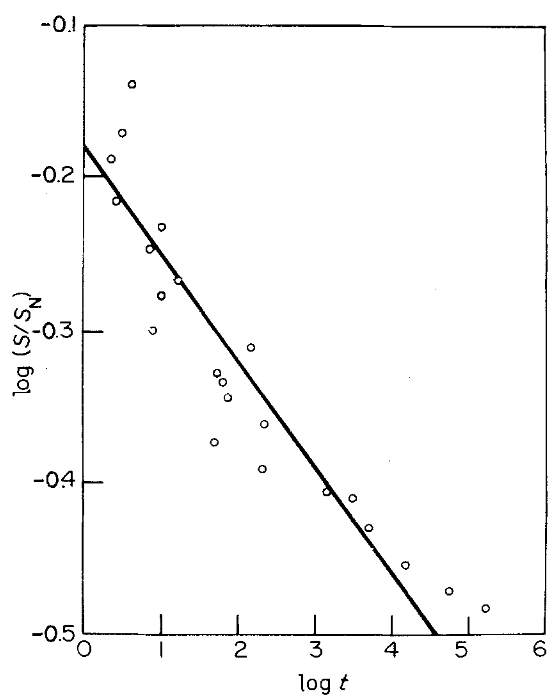

Figure 6 Fit of Equation 2 to mean $\log_{10}$ failure times of Pyrex borosilicate glass at different stresses.

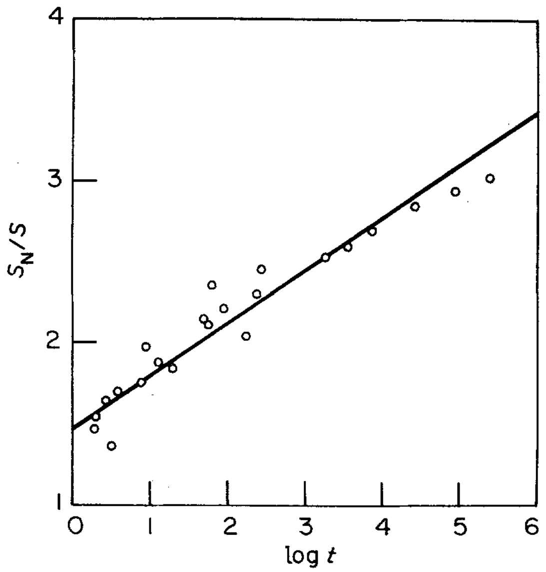

Figure 7 Fit of Equation 3 to mean $\log_{10}$ failure times of Pyrex borosilicate glass at different stresses.

The variation of the spread (standard deviation) of $\log_{10} t$ values with stresses is a sensitive measure of the functional dependence of failure time on stress [5]. The applied stress S can be considered constant compared to the spread in inert strengths (S_N values) for different specimens. Thus the differential of S/S_N is:

$$d(S/S_N) = -(S\, dS_N)/S_N^2. \tag{13}$$

If the differential of the $\log_{10} t$ is considered to equal the standard deviation of $\log_{10} t$ (designated by $\delta \log_{10} t$) and $dS_N/\bar{S}_N$ is the coefficient of variation (standard deviation divided by the mean) of the inert strength $(\delta S_N)/\bar{S}_N$, then the following relations are found from Equations 1 to 4:

$$\delta \log_{10} t = b \frac{S}{S_N} \frac{\delta S_N}{S_N}, \tag{14}$$

$$\delta \log_{10} t = n \frac{\delta S_N}{\bar{S}_N}, \tag{15}$$

$$\delta \log_{10} t = g \frac{\bar{S}_N}{S} \frac{\delta S_N}{\bar{S}_N}, \tag{16}$$

and

$$\delta \log_{10} t = 2p \left(\frac{\bar{S}_N}{S}\right) \frac{\delta S_N}{S_N}. \tag{17}$$

These equations show that for the direct dependence of Equation 1 the spread in $\log_{10} t$ values should decrease as the stress decreases, that for the power law of Equation 2 the spread in $\log_{10} t$ should remain constant with changes in stress, and

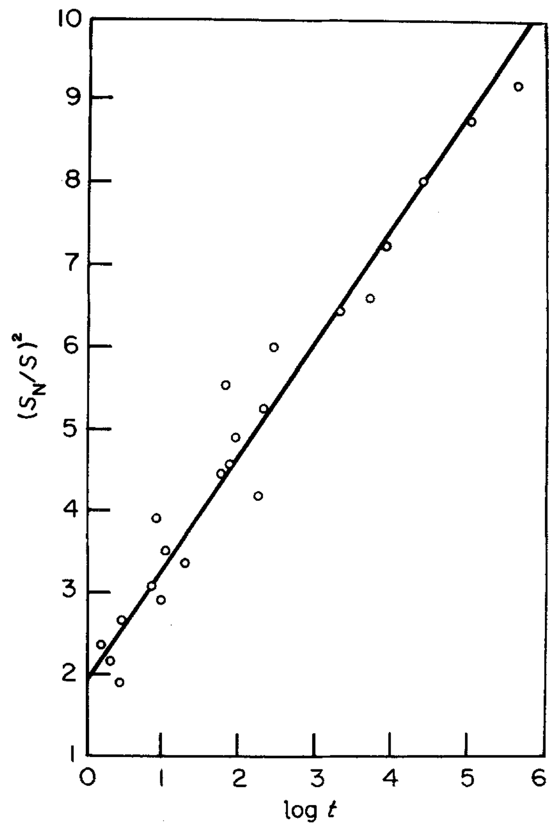

Figure 8 Fit of Equation 4 to mean $\log_{10}$ failure times of Pyrex borosilicate glass at different stresses.

for the inverse exponentials of Equations 3 and 4 the spread in $\log_{10} t$ should increase as the stress decreases. In these equations $\bar{S}_N$ is the mean value of strengths at -196° C.

For every set of static fatigue data on glass that are reliable enough, the spread in $\log_{10} t$ values increases as the stress decreases [5, 16, 17]. The same trend is evident in Table II for the present data. Therefore this test also favours Equations 3 and 4. The product $g(\delta S_N/\bar{S}_N)$ calculated from a regression analysis of the $\delta \log_{10} t$ values and Equation 16 is about 0.53, whereas the value of this product from the fit to Equation 3 and the -196° C fracture spread is 0.70. Similarly, the product $2p(\delta S_N/\bar{S}_N)$ calculated from the $\delta \log_{10} t$ values is about 0.24 and the value calculated from the fit to Equation 4 is 0.32. These comparisons show that the standard deviations in $\log_{10} t$ are consistent with either Equations 3 and 4, and cannot distinguish which equations fit best.

The predictions of sample life from Equations 1 to 4 were tested with samples at the low stress of $S/S_N = 0.253$. Twenty samples were started at this stress, with the following results: samples failed at 5, 9, 17, 55, 77 and 467 days; after 750 days of stressing seven samples were unbroken; and seven samples were removed from the test after 355 days. At first the samples removed from the test are not considered. Of the remaining thirteen samples, six failed at the times given above. These results can be plotted on probability paper, where they show a reasonable straight line with a mean $\log_{10}$ failure time of 7.8 (time in sec) or about 730 days and a standard deviation in $\log_{10} t$ of about 1.65. The samples removed from test at 355 days are consistent with these values. Predicted mean failure times from Equations 1 to 4 and the parameters in Table IV are given in Table V. The long-time failure tests are closest to the prediction of Equation 3. Of course this test involves only a small number of samples, so the comparison with expected failure times and standard deviation is only rough, and perhaps the comparison should be limited to a test of consistency. The predictions of Equations 1 and 2 are clearly inconsistent with the long-time results, giving expected life of about three and two orders of magnitude respectively, less than found experimentally.

5. Discussion

The present results show that either Equations 3 or 4 should be used for life prediction of glass parts in preference to Equations 1 and 2. More data and analysis are needed to decide definitely between these two relations. More parameters in these equations could give better fits to some of the data but such a step does not seem justified at present. The power law of Equation 2 can be used to extrapolate static fatigue data for only a short time beyond experimental times, in spite of the large number of calculations and interpretations of fatigue data in terms of this equation.

Hillig and Charles developed a theory of static fatigue of glass as resulting from the stress-enhanced rate of reaction of water with the silicate network of the glass [1]. The functional dependence of failure time on stress. Hillig and Charles assumed that the rate of reaction, v, of water with glass depended on the stress, σ, at the crack tip by the equation

$$v = v_0 \exp \beta\sigma, \tag{18}$$

where v_0 and β are constants, and found the relation of Equation 1 between failure time and stress. Their treatment can be modified [21] by using the equations

$$v = A\sigma^n, \tag{19}$$

and

$$v = v_\infty \exp(-\alpha/\sigma), \tag{20}$$

to obtain Equations 2 and 3, and the relationship

$$v = v_\infty \exp(-\alpha'/\sigma^2) \tag{21}$$

gives Equation 4. The present results suggest that either Equation 20 or 21 is the correct one. Both of these equations have theoretical support. Taylor [3] derived Equation 19 from a simple bond-stretching model of the influence of stress of a chemical reaction, and Gilman and Tong [22] found this equation from a tunneling model of fracture. Elliot [6] derived Equation 20 from an analogy of diffusion in oxide films. Determination of the exact mechanism by which water leads to fatigue in glass requires further study.

TABLE V Predicted failure times for $S/S_N = 0.253$

$f(S/S_N)$	Equation	Predicted $\log_{10} t_f$	Days to fail
S/S_N	1	4.84	0.80
$\log_{10}(S/S_N)$	2	5.90	9.20
S_N/S	3	7.36	265
$(S_N/S)^2$	4	9.31	23700

Acknowledgements

This work was supported by the U.S. Army Research Office. We would like to thank J. A. Malitoris for help with the experiments and for use of his data, and T. Bamford and S. Capella for help with the experiments.

Appendix: Weighted least squares

Consider a set of independent variables X_i (for example, stresses) and a set of corresponding dependent measured variables Y_i (for example mean failure times). Assume these variables are linearly related and calculate estimates of the parameters A and B in the linear relationship

$$\hat{Y} = A + BX \tag{A1}$$

where $\hat{Y}$ is the predicted value for a given X. The least-squares equations for A and B are

$$B = \frac{n\Sigma X_i Y_i - \Sigma X_i \Sigma Y_i}{n\Sigma X_i^2 - (\Sigma X_i)^2} \tag{A2}$$

and

$$A = \frac{\Sigma Y_i - B\Sigma X_i}{n}, \tag{A3}$$

where the sums are from $i = 1$ to $i = n$ and n is the total number of values of Y_i. If each value of Y_i has a weight w_i, the equations for B and A become [20]

$$B = \frac{\Sigma X_i Y_i \Sigma w_i - \Sigma w_i X_i - \Sigma w_i Y_i}{\Sigma w_i \Sigma X_i^2 - \Sigma X_i \Sigma w_i X_i} \quad \text{(A4)}$$

$$A = \frac{\Sigma X_i^2 w_i Y_i - \Sigma X_i Y_i \Sigma X_i}{\Sigma w_i \Sigma X_i^2 - \Sigma X_i \Sigma w_i X_i} \quad \text{(A5)}$$

These equations revert to A2 and A3 if all the w_i factors are the same. The weighted regression coefficient R^2 is:

$$R^2 = \frac{[\Sigma w_i \Sigma w_i X_i Y_i - \Sigma w_i Y_i \Sigma w_i X_i]^2}{[\Sigma w_i \Sigma w_i X_i^2 - (\Sigma w_i X_i)^2][\Sigma w_i \Sigma w_i Y_i^2 - (\Sigma w_i Y_i)^2]}. \quad \text{(A6)}$$

References

1. W. B. HILLIG and R. J. CHARLES, in "High Strength Materials" edited by V. F. Zackey (Wiley Interscience, New York, 1965) p. 682.
2. R. J. CHARLES, *J. Appl. Phys.* **29** (1958) 1544, 1554.
3. N. W. TAYLOR, *J. Appl. Phys.* **18** (1947) 943.
4. S. M. COX, *Phys. Chem. Glasses* **10** (1969) 226.
5. E. K. PAVELCHEK and R. H. DOREMUS, *J. Non-cryst. Solids* **20** (1976) 305.
6. H. A. ELLIOT, *J. Appl. Phys.* **29** (1958) 224.
7. G. S. FRIEDMAN, MSc thesis, Rensselaer Polytechnic Institute, Troy, New York, 1981.
8. C. M. KIM, MSc thesis, Rensselaer Polytechnic Institute, Troy, New York, 1978.
9. J. A. MALITORIS, MSc thesis, Rensselaer Polytechnic Institute, Troy, New York, 1979.
10. R. LANGLEY, "Practical Statistics" (Dover, New York, 1971) pp. 166–178.
11. G. W. MOREY, "The Properties of Glass" 2nd edn (Reinhold, New York, 1954) Chapt. VI, pp. 166–190.
12. J. H. SIMMONS, S. A. MILLS and A. NAPOLITANO, *J. Amer. Ger. Soc.*, **57(3)** (1974) 109.
13. M. TOMAZAWA and T. TAKAMORI, *ibid.* **60(7–8)** (1977) 301.
14. G. K. BHATTACHARYA and R. A. JOHNSON, "Statistical Concepts and Methods" (Wiley Interscience, New York, 1977) p. 210ff.
15. K. V. BURY, "Statistical Models in Applied Science", (Wiley Interscience, New York, 1975) p. 86.
16. K. JAKUS, D. C. COYNE and J. E. RITTER, *J. Mater. Sci.* **13** (1978 2071.
17. J. E. BURKE, R. H. DOREMUS, W. B. HILLIG and A. M. TURKALO, in "Ceramics in Severe Environments", edited by W. W. Krigel and H. Palmour (Plenum Press, New York, 1971) p. 435.
18. B. J. WILKINS, *J. Amer. Cer. Soc.* **54** (1971) 593.
19. *Idem, J. Mater. Sci.* **7** (1972) 251.
20. N. R. DRAPER and H. SMITH "Applied Regression Analysis" 2nd edn (Wiley Interscience, New York, 1981) pp. 77–81.
21. R. H. DOREMUS, *Eng. Fract. Mech.* **13** (1980) 945.
22. J. J. GILMAN and H. C. TONG, *J. Appl. Phys.* **42** (1971) 3479.

Received 30 June and accepted 18 August 1981